OU-HET 761-2012

The Planck length as a duality of the Cosmological Constant: S-dS and S-AdS thermodynamics from a single expression

Ivan Arraut
Department of Physics, Osaka University, Toyonaka, Osaka 560-0043, Japan

Abstract

In this paper we suggest that the Planck length l_{pl} and the Cosmological Constant scale $r_\Lambda = \frac{1}{\sqrt{\Lambda}}$ could in principle be dual each other if we take seriously the so-called q-Bargmann Fock space representation as has been previously suggested by Kempf and others and if additionally we introduce l_{pl} as an ultraviolet cut-off and $r_\Lambda = \frac{1}{\sqrt{\Lambda}}$ as an infrared one. As a consequence, it is possible to demonstrate that a Generalized Uncertainty Principle (GUP) given by $\Delta X \Delta P \geq \frac{\hbar}{2} + \frac{l_{pl}^2}{2\hbar}(\Delta P)^2 + \frac{\hbar}{2r_\Lambda^2}(\Delta X)^2$, can reproduce appropriately the thermodynamic behavior for both, the Schwarzschild Anti de-Sitter (S-AdS) and the Schwarzschild de-Sitter (S-dS) space without making any analytic extension for the coefficient (parameter) related to the minimum uncertainty in momentum (already suggested in the literature). This is possible if the Black Hole temperature is described with respect to the "natural" Static Observer for the S-dS case located at a distance $l_0 = \left(\frac{3}{2} r_s r_\Lambda^2\right)^{1/3}$.

PACS numbers: 04.70.Dy

arXiv:1205.6905v3 [gr-qc] 10 Sep 2012

I. INTRODUCTION

Every approach to Quantum Gravity agrees in the fact that at the Planck scale (l_{pl}) the structure of the spacetime is discrete [1–6]. The Generalized Uncertainty Principle (GUP) has been introduced as a way to understand partially what happens at that scale. It has been used for the possible modifications of the physics for systems such as the Harmonic Oscillator [7] and Black Hole Thermodynamics [8]. Most of the different analysis are performed by introducing an ultraviolet (UV) cut-off at the Planck scale [7]. Other authors also include an infrared (IR) cut-off by using the so-called q-Bargmann Fock formalism [8–10]. In such a case, the UV scale is in some sense dual to the IR one. From this formalism, GUP is written as $\Delta X \Delta P \geq 1 + l_{pl}^2(\Delta P)^2 + \frac{1}{r_\Lambda^2}(\Delta X)^2$ ($\hbar = 1$) if we impose the l_{pl} as UV cut-off and the Cosmological Constant $\Lambda = \frac{1}{r_\Lambda^2}$ as an IR one. The introduction of Λ in the formalism of GUP [11–13] has already been studied in the literature. In [11], the derivation was heuristic and following the analogies suggested in [8]. In such a case, GUP was written with ΔX as a function of ΔP and assuming the validity of the condition $\Delta P \approx \frac{1}{\Delta X}$, which is related to the negative heat capacity for Black Holes [8, 11, 13]. Under this approximation we observe that the proposal given in [11] is just equivalent to that suggested in [12] for the S-dS case if the parameter related to the IR cut-off is said to be analytically continued into the imaginary plane. There are however some inconsistences if we simply make an analytic continuation. In [12] for example, as $\Delta X >> l_{pl}$, GUP was written for the S-dS case as $\Delta P \geq \frac{1}{\Delta X} - \frac{1}{r_\Lambda^2}\Delta X$. Although Λ is supposed to be introduced as an IR cut-off, it is clear that the previous expression simply gives $\Delta P_{min} = 0$ as $\Delta X = r_\Lambda$. But in Quantum Mechanics language this means that it would be possible to define exactly the momentum of a particle and still have a finite uncertainty in position. Additionally, if we accept the analogy suggested by Adler and Santiago [8], namely $\Delta P \approx \kappa$ (the surface gravity) and $\Delta X \approx r_H$ (the event horizon); then the result obtained in [12] is clearly in contradiction with the minimum temperature found by Bousso and Hawking for the S-dS Black Hole thermodynamics [14]. On the other hand, the approach followed in [11] has the problem that it was assumed that $\Delta X = r_H \approx 2GM$. But the event horizon for the S-dS black hole is not exactly given by $2GM$, this is just an approximation valid as the Black Hole mass satisfies the condition $M << M_{max} = \frac{1}{3}\frac{m_{pl}^2}{m_\Lambda}$ [11]. Additionally, it was not clear how to fix the parameter related to the IR cut-off and the value taken by that parameter fix the minimum temperature. In [11] for example, GUP is written as $\Delta X \geq \frac{1}{2\Delta P} - \frac{\gamma}{3}\frac{m_\Lambda^2}{(\Delta P)^3}$, where $r_\Lambda = \frac{1}{m_\Lambda}$ and it was used the validity of the approximation $\Delta X \approx \frac{1}{\Delta P}$ (negative heat capacity condition for BH). In such a case, GUP could be written as $\Delta P \geq \frac{1}{2\Delta X} - \frac{\gamma}{3}\frac{m_\Lambda^2}{(\Delta P)^2 \Delta X}$ or $\Delta P \geq \frac{1}{2\Delta X} - \frac{\gamma'}{3}m_\Lambda^2(\Delta X)$. If for example $\gamma' = \frac{3}{2}$, then $\Delta P_{min} = 0$ as $\Delta X_{max} = r_\Lambda$. We have then some degree of arbitrariness for the GUP expression suggested in [11].

In the case of S-AdS, it is already known that there is also a minimum temperature for the Black Hole [15]. This temparature is obtained as the Black Hole has a mass given by $M_{crit} = \frac{2}{3}\frac{m_{pl}^2}{m_\Lambda}$. If the mass is bigger or smaller than this value, then the Black Hole temperature increases. Then the S-AdS Black Hole has a negative heat capacity as $M < M_{crit}$ and a positive heat capacity one as $M > M_{crit}$. It is trivial to show that if $\Delta X = r_+$ (here we will denote r_+ as the event horizon for the S-AdS space) and $\Delta P = \kappa$, GUP with UV and IR cut-offs, reproduces the appropriate results for S-AdS without any problem or contradiction. This is possible since GUP given by $\Delta X \Delta P \geq 1 + l_{pl}^2(\Delta P)^2 + \frac{1}{r_\Lambda^2}(\Delta X)^2$ has

four solutions. Two of them complement providing the negative heat capacity region for the Black Hole temperature with a minimum and maximum temperature. This portion agrees for both, de-Sitter and Anti de-Sitter spaces. The other two solutions represent the positive heat capacity region for the AdS black hole and the transplanckian (modes) temperatures respectively. For the AdS case, the event horizon increases with the mass [15]. The two solutions with negative Heat Capacity are joined at the UV-IR mix scale, defined as the geometric average of the l_{pl} scale and the Λ one ($l_0 = (l_{pl} r_\Lambda)^{1/2}$). This scale represents an extremal condition for GUP. At this scale the uncertainty principle (in generalized form) takes its minimum value.

In this paper we demonstrate that GUP with UV and IR cut-offs, can in principle explain in a satisfactory way the Thermodynamic associated to the de-Sitter space without making any analytic continuation (changing the sign of Λ) for the parameter related to the minimum uncertainty in momentum. This is possible if the temperature for the S-dS case is defined with respect to the "natural" static observer position $\left(r_g = \left(\frac{3}{2} r_s r_\Lambda^2\right)^{1/3}\right)$ defined as the distance at which the attractive effects due to gravity and the repulsive ones due to Λ just cancel.

The paper is organized as follows: In Sections II and III, we make a brief review for the standard de-Sitter Black Hole thermodynamics as has been already performed by Bousso and Hawking [14]. In section IV, we make a brief review of GUP with UV cut-off as has been already performed by Adler, Santiago and others. In section V we introduce the formalism developed by Kempf and we suggest the l_{pl} scale as an UV cut-off and the Λ scale as an IR one, we then make the formal derivation of the IR-UV mix scale given given by $l_0 = (l_{pl} r_\Lambda)^{1/2}$ as an extremal condition for GUP. In section VI, we solve GUP in order to obtain the different solutions of the quadratic GUP equation. In section VII, we explain how can be obtained the de-Sitter Black Hole temperature when the observations are defined by the "natural" Static Observer r_g. In Section VIII, we make a brief review for the AdS thermodynamic already performed by Hawking and Page [15] and we explain how the different solutions for GUP obtained in section VI are related to it. In section IX, we make a comparison between the dS Black Hole temperature and the AdS one and we explain the similarities when the dS thermodynamic is analyzed with respect to the "natural" Static Observer. And finally, in section X we conclude.

II. HORIZONS IN A DE-SITTER BLACK HOLE

As a starting point, we will first derive the event horizons corresponding to the Black Hole in an asymptotically de-Sitter space. The Schwarzschild de-Sitter (SD) metric is given by [16]:

$$ds^2 = -V(r)dt^2 + V(r)^{-1}dr^2 + r^2 d\Omega_2^2 \tag{1}$$

where:

$$V(r) = 1 - \frac{r_s}{r} - \frac{1}{3}\frac{r^2}{r_\Lambda^2} \qquad r_s = 2G_N M \qquad r_\Lambda = \frac{1}{\sqrt{\Lambda}} \tag{2}$$

the event horizons for this metric can be calculated by using the standard condition [16]:

$$g^{rr}(r_c) = 0 \tag{3}$$

the two event horizons corresponding to this space are given by [16]:

$$r_{CH} = -2r_\Lambda cos\left(\frac{1}{3}\left(cos^{-1}\left(\frac{3r_s}{2r_\Lambda}\right) + 2\pi\right)\right) \tag{4}$$

and:

$$r_{BH} = -2r_\Lambda cos\left(\frac{1}{3}\left(cos^{-1}\left(\frac{3r_s}{2r_\Lambda}\right) + 4\pi\right)\right)$$

where r_{CH} corresponds to the Cosmological Horizon and r_{BH} corresponds to the Black Hole event horizon. The equations 4 show that the maximum mass for a Black Hole in an universe with a positive Cosmological Constant Λ is given by:

$$M_{max} = \frac{1}{3}\frac{m_{pl}^2}{m_\Lambda} \tag{5}$$

where m_{pl} corresponds to the Planck mass and $m_\Lambda = \sqrt{\Lambda}$. If the mass of a Black Hole is bigger than the value 5, then there is no radiation process at all and we only have a naked singularity. M_{max} is however of the order of magnitude of the mass of the Universe if we take the observed value for Λ [11]. As $M = M_{max}$, the two event horizons take the same value $\left(r_{BH} = r_{CH} = r_\Lambda = \frac{1}{\sqrt{\Lambda}}\right)$, they are degenerate and there is no net radiation due to the thermodynamic equilibrium established. For degenerate horizons, the Schwarzschild-like coordinates given by the expression 1 and 2 are not valid anymore [14, 17]. As Bousso and Hawking have explained before [14, 17], as $M \to M_{max}$, then $V(r) \to 0$ between the two horizons (BH and Cosmological). In such a case we need a new coordinate system. In agreement with Ginsparg and Perry [18], we can write:

$$9G^2M^2\Lambda = 1 - 3\epsilon^2 \qquad 0 \leq \epsilon \ll 1 \tag{6}$$

Where the degenerate case (where the two horizons become the same) corresponds to $\epsilon \to 0$. We must then define the new radial and the new time coordinates to be:

$$\tau = \frac{1}{\epsilon\sqrt{\Lambda}}\psi \qquad r = \frac{1}{\sqrt{\Lambda}}\left(1 - \epsilon cos\chi - \frac{1}{6}\epsilon^2\right) \tag{7}$$

in these coordinates, the Black Hole horizon corresponds to $\chi = 0$ and the Cosmological horizon to $\chi = \pi$ [14, 17]. The metric then becomes:

$$ds^2 = -r_\Lambda^2\left(1 + \frac{2}{3}\epsilon cos\chi\right)sin^2\chi d\psi^2 + r_\Lambda^2\left(1 - \frac{2}{3}\epsilon cos\chi\right)d\chi^2 + r_\Lambda^2(1 - 2\epsilon cos\chi)d\Omega_2^2 \tag{8}$$

this metric has been expanded up to first order in ϵ. Eq. 8 is of course the appropriate metric to be used as the mass of the Black Hole is near to its maximum value given by 5. The importance of the previous analysis will become clear in the next section.

III. THE MINIMUM TEMPERATURE FOR A BLACK HOLE IN A SCHWARZSCHILD-DE SITTER (SD) SPACE

In general, the surface gravity, can be calculated as [14, 17]:

$$\kappa_{BH,CH} = \left(\frac{(K^\mu \nabla_\mu K_\gamma)(K^\alpha \nabla_\alpha K^\gamma)}{-K^2} \right)^{1/2}_{r=r_{BH,CH}} \tag{9}$$

The subscripts BH and CH correspond to the Black Hole and Cosmological Horizons respectively. The spacetime described by the metric of the previous section, admits a timelike Killing vector field. It is given by:

$$K = \gamma_t \frac{\partial}{\partial t} \tag{10}$$

the surface gravity in general depends on the selected normalization for the Killing vector field. In other words, it depends on the choice of γ_t. This is a consequence of the observer dependent notion of the Black Hole temperature[19] which is related to the static observer position. In an asymptotically flat spacetime, the "natural" static observer is at infinity. In such a case, the normalization of the Killing vector is simply $K^2 = -1$ ($\gamma_t = 1$) at infinity. This normalization cannot be used in general when the spacetime is not asymptotically flat. In the case of an asymptotically de-Sitter spacetime for example, there exist a Cosmological Horizon of the order of magnitude of r_Λ. In this spacetime (S-dS), $\gamma_t \approx 1$ only as the condition $r_s << r_\Lambda$ is satisfied.
We will define the "natural" static observer position as r_g. At this distance, the orbit of the Killing vector coincides with the geodesic through r_g at constant angular variables [14, 17]. In other words, we will take it to be the S^2 sphere at which the attractive effects due to gravitation are cancelled exactly with the repulsive ones produced by Λ.
An observer at r_g does not feel any acceleration, he is in an unstable equilibrium. This scale is obtained from the condition [16]:

$$\frac{dg_{00}}{dr} = 0 \tag{11}$$

this condition for the S-dS metric gives the result:

$$r_g = \left(\frac{3}{2} r_s r_\Lambda^2 \right)^{1/3} \tag{12}$$

with this position for the static observer, the normalization factor for the Killing vector becomes [14, 17]:

$$\gamma_t = U(r_g)^{1/2} = \left(1 - \left(\frac{3r_s}{2r_\Lambda} \right)^{2/3} \right)^{1/2} \tag{13}$$

and the equation for the surface gravity given in 9, takes the form:

$$\kappa_{CH,BH} = \frac{1}{2\sqrt{U(r_g)}} \left| \frac{\partial U}{\partial r} \right|_{r=r_BH,CH} \tag{14}$$

where for the S-dS metric we get:

$$\kappa_{BH,CH} = \frac{\left|\frac{1}{r_H} - \frac{r_H}{r_\Lambda^2}\right|}{2\left(1 - \left(\frac{3r_s}{2r_\Lambda}\right)^{2/3}\right)^{1/2}} \tag{15}$$

here we use r_H for the Black Hole event horizon or for the Cosmological one. The absolute value is necessary if we want to guarantee a positive definite temperature for both, the Black Hole Horizon and the Cosmological one. Physically, it is due to the fact that for the static observer; any object below his position-namely $r < r_g$ will move toward the Black Hole horizon. On the other hand, any object at distances larger that r_g, will move toward the Cosmological Horizon. In the neighborhood of the maximum mass given by 5, as we have explained before, the Schwarzschild like coordinates become inappropriate. In agreement with Bousso and Hawking, the surface gravity at this regime is given by:

$$\kappa_{CH,BH} = \frac{1}{r_\Lambda}\left(1 \mp \frac{2}{3}\epsilon\right) + O(\epsilon^2) \tag{16}$$

corresponding to the Black Hole (+ sign) and the Cosmological Horizon (- sign) respectively. ϵ is already defined in 6. The minimum temperature for a Black Hole in an asymptotically de-Sitter space is then given by:

$$\kappa_{min}^{BH} = \frac{1}{r_\Lambda} \tag{17}$$

For the Cosmological Horizon, the minimum temperature is obtained as $M = 0$ and it is given by $\kappa_{min}^{CH} = \frac{1}{\sqrt{3}}\frac{1}{r_\Lambda}$, which is smaller than the value 17 but of the same order of magnitude. The Cosmological Horizon has a positive Heat Capacity. In other words, it is stable. We must take into account that the Cosmological Horizon decreases with the mass. This statement becomes important at the moment of making an analogy with the Generalized Uncertainty Principle (GUP) as we will explain later in this paper.

IV. ULTRAVIOLET (UV) GENERALIZED UNCERTAINTY PRINCIPLE AND BLACK HOLE THERMODYNAMICS

In this section we will briefly summarize the approach to the Black Hole Thermodynamics via GUP. In this case, we will assume that there is no IR cut-off. Only the UV cut-off will be taken into account. In such a case, GUP provides the following expression [8]:

$$\Delta X \Delta P \geq \frac{1}{2} + \frac{l_{pl}^2}{2}\Delta P^2 \tag{18}$$

if we take the uncertainty in position as the Black Hole event horizon $\Delta X \approx r_H$ and the momentum uncertainty as the surface gravity $\Delta P \approx \kappa$ [8], then we obtain:

$$\kappa \approx \frac{1}{2r_H} + l_{pl}^2\frac{\kappa^2}{2r_H} \tag{19}$$

this is just a quadratic equation for the surface gravity. If we solve this equation, then we obtain:

$$\kappa_{1,2} = \frac{r_H}{l_{pl}^2}\left(1 \mp \sqrt{1 - \left(\frac{l_{pl}}{r_H}\right)^2}\right) \tag{20}$$

where the positive sign corresponds to a positive heat capacity solution (κ_2) and the negative sign corresponds to the negative heat capacity condition for the standard Black Hole (κ_1). In agreement with the expression 20, there is a Black Hole remnant which is obtained as the event horizon takes the value $r_H = l_{pl}$. In such a case, the surface gravity takes its maximum (minimum) value given by:

$$\kappa_{max1} = \kappa_{min2} = \frac{1}{l_{pl}} \tag{21}$$

on the other hand, as $r_H >> l_{pl}$, equation 20 gives the following results:

$$\kappa_1 \approx \frac{1}{2r_H} \qquad \kappa_2 \approx \frac{2r_H}{l_{pl}^2} \tag{22}$$

where κ_1 corresponds to the Schwarzschild Black Hole solution and κ_2 would correspond to the transplanckian modes for the temperature with a positive heat capacity region which will not be considered in this manuscript. If $r_H \approx 2GM$ at this regime, then it is easy to verify that κ_1 is just the Schwarzschild Black Hole surface gravity. Take for example the limit $r_\Lambda \to \infty$ in the expression 15. In such a case, you will just recover the result 22 for κ_1 again.

V. ULTRAVIOLET-INFRARED (UV-IR) GENERALIZED UNCERTAINTY PRINCIPLE (GUP) AND BLACK HOLE THERMODYNAMICS

If we take seriously the formalism developed by Kempf [9, 10] about the minimum Uncertainties in Position and Momenta, then we can write GUP in a more symmetric formulation with respect to position and momentum as:

$$\Delta X \Delta P \geq \frac{\hbar}{2}\left(1 + (q^2 - 1)\left(\frac{(\Delta X)^2}{4L^2} + \frac{(\Delta P)^2}{4K^2}\right)\right) \tag{23}$$

here we have assumed $< X >=< P >= 0$ [9, 10]. In agreement with the formalism developed in [9, 10], the minimum scale in position corresponds to:

$$\Delta X_{min} = L\sqrt{1 - q^{-2}} \tag{24}$$

additionally, it is known that the smallest uncertainty in momentum is given by:

$$\Delta P_{min} = K\sqrt{1 - q^{-2}} \tag{25}$$

K and L must satisfy the additional constraint:

$$KL = \frac{(q^2 + 1)\hbar}{4} \tag{26}$$

this condition suggests that the UV scale is dual to the IR one. This formulation has two free parameters, which we will fix in agreement with the Planck scale l_{pl} and the Cosmological

Constant scale $\Lambda = \frac{1}{r_\Lambda^2}$. We want an expression for GUP symmetric with respect to position and momentum given by:

$$\Delta X \Delta P \geq \frac{\hbar}{2} + \frac{l_{pl}^2}{2\hbar}(\Delta P)^2 + \frac{\hbar}{2r_\Lambda^2}(\Delta X)^2 \tag{27}$$

if we want this expression to agree with 23, the following conditions must be satisfied:

$$l_{pl} = L\sqrt{1-q^{-2}} \tag{28}$$

$$\frac{1}{r_\Lambda} = K\sqrt{1-q^{-2}}$$

these expressions together with the condition 26, automatically fix the value of q. Although q could take different values in order to satisfy the previous conditions, we select the value of q satisfying the additional constraint $q \geq 1$ [9, 10, 20]. In such a case, q, which is supposed to be related to the gravitational degrees of freedom, satisfies:

$$q \approx 1 + \frac{l_{pl}}{r_\Lambda} \tag{29}$$

where $l_{pl} \approx 10^{-35}mt$ is the Planck scale and $r_\Lambda \approx 10^{26}mt$ is the Hubble one, then $q \approx 1 + 10^{-61}$. If $\Lambda \to 0$ then $q = 1$ and $l_{pl} \to 0$ and vice versa. It means that inside this formalism, the Cosmological Constant Λ is related to the minimum scale in position. Without a minimum scale, there is no Cosmological Constant and vice versa. Originally Kempf [9, 10] derived the results 24 and 25 by definiting the function:

$$f(\Delta X, \Delta P) := \Delta X \Delta P - \frac{\hbar}{2}\left(1 + (q^2-1)\left(\frac{(\Delta X)^2 + <X>^2}{4L^2} + \frac{(\Delta P)^2 + <P>^2}{4K^2}\right)\right) \tag{30}$$

here, we will assume $<X> = 0 = <P>$. The minimum scale in position can be found given the following extremal condition:

$$\frac{\partial}{\partial \Delta P} f(\Delta X, \Delta P) = 0 \qquad f(\Delta X, \Delta P) = 0 \tag{31}$$

then, the result is just the eq. 28 $\left(\Delta X_{min} = L\sqrt{1-q^{-2}}\right)$. On the other hand, the minimum scale in momentum is obtained from the condition:

$$\frac{\partial}{\partial \Delta X} f(\Delta X, \Delta P) = 0 \qquad f(\Delta X, \Delta P) = 0 \tag{32}$$

the result is just the eq. 28 $\left(\Delta P_{min} = K\sqrt{1-q^{-2}}\right)$. We can however, derive a third scale given by the UV-IR mix effects. This scale was introduced for first time by John A. Wheeler [1] in 1957. It is given by the geometrical average of the l_{pl} and r_Λ, namely, $l_0 = (l_{pl} r_\Lambda)^{1/2}$. We can define the total differential for the function $f(\Delta X, \Delta P)$ as:

$$df(\Delta X, \Delta P) = \left(\frac{\partial f(\Delta X, \Delta P)}{\partial \Delta P}\right)_{\Delta X = C} d(\Delta P) + \left(\frac{\partial f(\Delta X, \Delta P)}{\partial \Delta X}\right)_{\Delta P = C} d(\Delta X) \tag{33}$$

The general extremal condition inside the phase space is obtained as the total differential 33 goes to zero. In such a case:

$$df(\Delta X,\Delta P)=0 \tag{34}$$

obtaining then the result:

$$\frac{d(\Delta X)}{d(\Delta P)}=-\frac{\Delta X}{\Delta P}\frac{\left(1-\frac{\hbar}{4K^2}(q^2-1)\frac{\Delta P}{\Delta X}\right)}{\left(1-\frac{\hbar}{4L^2}(q^2-1)\frac{\Delta X}{\Delta P}\right)} \tag{35}$$

let's impose now the constraint $f(\Delta X,\Delta P)=0$. In such a case 30 becomes:

$$\Delta X\Delta P=\frac{\hbar}{2}\left(1+(q^2-1)\left(\frac{(\Delta X)^2+<X>^2}{4L^2}+\frac{(\Delta P)^2+<P>^2}{4K^2}\right)\right) \tag{36}$$

if we differentiate with respect to ΔX we have:

$$\frac{d}{d(\Delta X)}(\Delta X\Delta P)=\Delta P+\Delta X\frac{d(\Delta P)}{d(\Delta X)}=\frac{\hbar}{4}\left((q^2-1)\left(\frac{\Delta X}{L^2}+\frac{\Delta P}{K^2}\frac{d(\Delta P)}{d(\Delta X)}\right)\right) \tag{37}$$

this equation is just the condition 34 as the reader can easily verify. The same result is obtained when the differentiation is done with respect to ΔP. In such a case:

$$\frac{d}{d(\Delta P)}(\Delta X\Delta P)=\Delta X+\Delta P\frac{d(\Delta X)}{d(\Delta P)}=\frac{\hbar}{4}\left((q^2-1)\left(\frac{\Delta X}{L^2}\frac{d(\Delta X)}{d(\Delta P)}+\frac{\Delta P}{K^2}\right)\right) \tag{38}$$

which is just equivalent to the results 34 and 37. In fact, the condition 34 is just equivalent to:

$$2d(\Delta X\Delta P)=\frac{\hbar}{4}d\left((q^2-1)\left(\frac{(\Delta X)^2}{L^2}+\frac{(\Delta P)^2}{K^2}\right)\right) \tag{39}$$

then the condition $f(\Delta X,\Delta P)=0$ is just included inside 34. We want to know what is the extremal condition for the Uncertainty Principle $\Delta X\Delta P$. We want to know when $\frac{d(\Delta X\Delta P)}{d\Delta X}=0=\frac{d(\Delta X\Delta P)}{d\Delta P}$. In such a case, from 37 it is clear that we have to satisfy:

$$\frac{\Delta X}{\Delta P}=-\frac{d(\Delta X)}{d(\Delta P)} \tag{40}$$

introducing this result inside the right-hand side of 37 or 38, we obtain:

$$(q^2-1)\left(\frac{\Delta X}{L^2}+\frac{\Delta P}{K^2}\frac{d(\Delta P)}{d(\Delta X)}\right)=0 \tag{41}$$

this equation has two solutions. The first one is not interesting for us, because it suggests $q=1$ which is a trivial condition because in such a case $d(\Delta X\Delta P)=0$ everywhere. Additionally, $q=1$ corresponds to the standard Bosonic algebra in agreement with [9, 10, 16]. We then do not consider that case here. The interesting case is:

$$\left(\frac{\Delta X}{L^2}+\frac{\Delta P}{K^2}\frac{d(\Delta P)}{d(\Delta X)}\right)=0 \tag{42}$$

which in combination with 40 gives:

$$\Delta X = \pm \frac{L}{K} \Delta P \tag{43}$$

if we compare the expressions 27 with the one obtained in 36 under the condition $f(\Delta X, \Delta P) = 0$, then it is simple to verify that ($\hbar = 1$):

$$L = K^{-1} = \frac{\sqrt{2}}{2} (l_{pl} r_\Lambda)^{1/2} \tag{44}$$

then the condition 43 becomes:

$$\Delta X = \frac{1}{2} (l_{pl} r_\Lambda) \Delta P \tag{45}$$

here we have only taken into account the positive sign. For consistence, it is easy to verify that the condition 26 is satisfied. It can be verified if we take into account that in agreement with 29, we have $q^2 \approx 1 + 2\frac{l_{pl}}{r_\Lambda}$. Then 26 becomes ($\hbar = 1$):

$$KL \approx \frac{1}{2} \tag{46}$$

This result is consistent with 44 as can be verified. If we replace 45 inside 27, we then obtain under the approximation $r_\Lambda >> l_{pl}$, the following result:

$$\Delta X \approx (l_{pl} r_\Lambda)^{1/2} = l_0 \qquad \Delta P \approx \frac{1}{(l_{pl} r_\Lambda)^{1/2}} = \frac{1}{l_0} \tag{47}$$

The result 47 is just the UV-IR scale already defined in [5] and obtained for first time by John A. Wheeler in 1957 [1] and interpreted as a coherence region.

VI. THE MINIMUM SCALE IN POSITION AND MOMENTUM AND BLACK HOLE THERMODYNAMICS

Here we want to show that in principle it is possible to reproduce the S-dS Thermodynamics with an UV cut-off without making any analytic extension for the IR cut-off scale in the GUP expression. In order to find the behavior of ΔX or ΔP near their minimum values, we must solve for ΔX near the r_Λ scale and for ΔP near the l_{pl} scale. The GUP expression solved for ΔX is valid as $\frac{1}{\sqrt{l_{pl} r_\Lambda}} \geq \Delta P \geq \frac{1}{r_\Lambda}$. On the other hand, the GUP expression solved for ΔP is valid as $(l_{pl} r_\Lambda)^{1/2} \geq \Delta X \geq l_{pl}$. Both solutions are equal at the UV-IR mix scale given by $\Delta X = l_0 = (l_{pl} r_\Lambda)^{1/2}$. Note that the GUP equation 27 is quadratic for both, position and momentum. We then expect two solutions of ΔX and two more for the solutions with respect to ΔP. We want to choose the solutions such that as $\Delta X >> l_{pl}$ and $\Delta P >> \frac{1}{r_\Lambda}$ we can then guarantee a negative heat capacity behavior for the Black Hole event horizon in agreement with the analogy suggested by Chen, Adler and Santiago [8]. In such a case, $\Delta X = r_H$ and $\Delta P = \kappa$ correspond to the event horizon and the surface gravity respectively. One of the lessons learned from the S-dS metric is that we cannot say in this analogy that $\Delta X \approx 2GM$. This is just an approximation valid as $M << M_{max}$ or $\Delta X << r_\Lambda$ in agreement with eq. 5 and [11]. For $\hbar = 1$, let's solve for $\Delta X = r_H$ in equation 27. In such a case, we obtain:

$$\Delta X_1 \approx \left(r_\Lambda^2 \Delta P_1\right)\left(1-\sqrt{1-\frac{1}{(\Delta P_1 r_\Lambda)^2}}\right) \tag{48}$$

as $\Delta P_1 >> \frac{1}{r_\Lambda}$, the condition $\Delta X_1 \approx \frac{1}{\Delta P_1}$ is satisfied. This is what is needed if we want to satisfy the negative heat capacity condition for Black Holes. On the other hand, in the neighborhood of l_{pl}, it is better to solve for ΔP. In such a case, we have:

$$\Delta P_1 \approx \left(\frac{\Delta X_1}{l_{pl}^2}\right)\left(1-\sqrt{1-\frac{l_{pl}^2}{(\Delta X_1)^2}}\right) \tag{49}$$

as $\Delta X_1 >> l_{pl}$, we recover again the standard result $\Delta P_1 \approx \frac{1}{\Delta X_1}$. The solutions 48 and 49 both satisfy the ordinary Uncertainty principle as $\Delta P_1 >> \frac{1}{r_\Lambda}$ and $\Delta X_1 >> l_{pl}$. They describe the same solution at different scale regimes of eq. 27. Eq. 48 is the solution valid when the condition $\frac{1}{\sqrt{l_{pl} r_\Lambda}} \geq \Delta P_1 \geq \frac{1}{r_\Lambda}$ is satisfied. On the other hand, 49 is the solution valid when $(l_{pl} r_\Lambda)^{1/2} \geq \Delta X_1 \geq l_{pl}$ is satisfied. The scale at which 48 and 49 provide the same result is:

$$\Delta X_1 = \sqrt{l_{pl} r_\Lambda} \qquad \Delta P_1 = \frac{1}{\sqrt{l_{pl} r_\Lambda}} \tag{50}$$

equation 50 shows the scale at which the UV modes are perfectly mixed with the IR ones. We have already derived formally this scale in 47. There are still two more solutions related to 48 and 49. They correspond to the positive sign solution for the quadratic equations. Let's rewrite the complement solutions to 48 and 49. They are:

$$\Delta X_2 = \left(r_\Lambda^2 \Delta P_2\right)\left(1+\sqrt{1-\frac{1}{(\Delta P_2 r_\Lambda)^2}}\right) \tag{51}$$

and:

$$\Delta P_2 = \left(\frac{\Delta X_2}{l_{pl}^2}\right)\left(1+\sqrt{1-\frac{l_{pl}^2}{(\Delta X_2)^2}}\right) \tag{52}$$

as $\Delta P_2 >> \frac{1}{r_\Lambda}$, then 51 becomes $\Delta X_2 \approx r_\Lambda^2 \Delta P_2$. In Black Hole thermodynamics, this corresponds to the positive heat capacity region for the surface gravity in the case of S-AdS Black Hole. We will explain this point in detail in future sections. On the other hand, as $\Delta X_2 >> l_{pl}$, the solution 52 becomes $\Delta P_2 \approx \frac{\Delta X_2}{l_{pl}^2}$. This solution will not be considered here since it is related to the transplanckian modes.

VII. INTERPRETATION

From the result 48, it is clear that there exist a minimum scale in momentum given by $\Delta P_{1min} = \frac{1}{r_\Lambda}$. The corresponding maximum position scale is $\Delta X_1 = r_\Lambda$. This is consistent with the results obtained in 4 and 17 in agreement with the S-dS Black Hole as it is described by the "natural" static observer. Then GUP with an infrared cut-off given by Λ, can describe

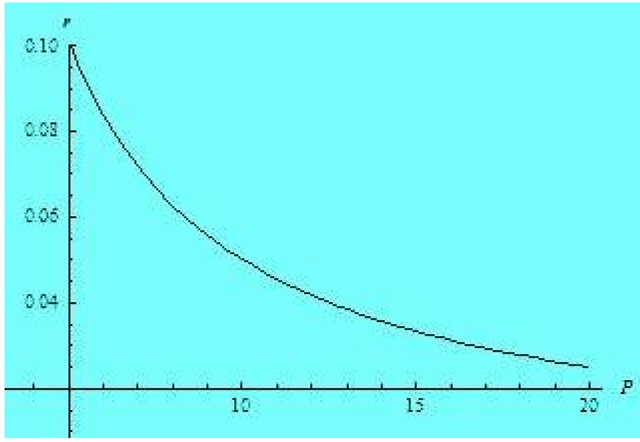

FIG. 1: Distance vs Momentum in agreement with the solution 48. Λ has been normalized to unity for convenience.

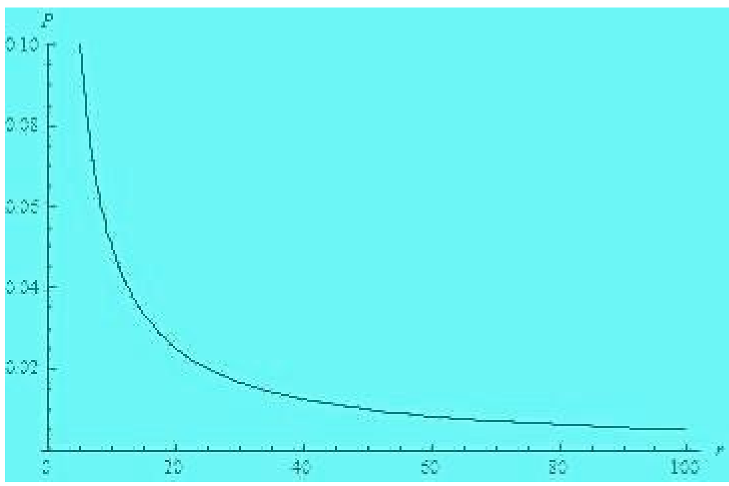

FIG. 2: Momentum vs distance in agreement with the solution 49. l_{pl} has been normalized to unity for convenience. It is evident the negative heat capacity behavior

the same results as in the case of the S-dS Black Hole Thermodynamics related to the Black Hole event Horizon as far as $r_H >> l_{pl}$. We identify $\Delta X_1 = r_H$ and $\Delta P_1 = \kappa$. In Fig. 1 we can see the plot of $\Delta X_1 = r$ vs $\Delta P_1 = P$ for the regime $(l_{pl} r_\Lambda)^{1/2} \leq \Delta X_1 \leq r_\Lambda$ or $\frac{1}{\sqrt{l_{pl} r_\Lambda}} \geq \Delta P_1 \geq \frac{1}{r_\Lambda}$. Fig 2 corresponds to eq. 49. It is just the continuation of the solution 48 but in the regime $l_{pl} \leq \Delta X_1 \leq (l_{pl} r_\Lambda)^{1/2}$ or $\frac{1}{l_{pl}} \geq \Delta P_1 \geq \frac{1}{(l_{pl} r_\Lambda)^{1/2}}$. Near the Planck scale the behavior deviates from the standard Black Holes in S-dS space. The S-dS thermodynamics is incomplete if we cannot reproduce the surface gravity for the Cosmological Horizon. The surface gravity in S-dS metric can be described with respect to the "natural" static observer in agreement with Bousso and Hawking [15]. In GUP language, this is equivalent to a redefinition of the UV and IR cut-offs for the Cosmological Horizon. If the "natural" static observer defines the same origin of coordinates for the Black Hole event Horizon and the Cosmological one, then he must redefine the cut-offs in order to make a description about the surface gravity for the Cosmological Horizon. If the BH event horizon is $r^0_{BH} = l_{pl}$, then the Cosmological Horizon is approximately $r^0_{CH} \approx \sqrt{3} r_\Lambda$ and the initial position for the

Static observer is $r_{0g} = \left(\frac{3}{2} l_{pl} r_\Lambda^2\right)^{1/3}$.
In agreement with the "natural" static observer, the IR cut-off for the Cosmological Horizon is given by $\Delta X_1^0 \approx \sqrt{3} r_\Lambda$, which corresponds to $\Delta P_{1min}^{CH} \approx \frac{1}{\sqrt{3}} \frac{1}{r_\Lambda}$. This is just the minimum value of the surface gravity for the Cosmological Horizon already found in section III for the standard S-dS Black Hole. If the static observer defines a different IR scale for the Cosmological Horizon, he must also define a new UV scale. The new UV for the static observer point of view is given by $\Delta X_{1min}^{CH} \approx r_\Lambda$ corresponding to $\Delta P_{1max}^{CH} = \frac{1}{r_\Lambda}$. This conditions and the appropriate behavior for the Cosmological Horizon surface gravity can be obtained from the equations 48 and 49 if we replace $r_\Lambda \to \sqrt{3} r_\Lambda$ and $l_{pl} \to r_\Lambda$. In such a case, the corresponding equations describing the Cosmological Horizon surface gravity are:

$$\Delta X_1 \approx \left(3 r_\Lambda^2 \Delta P_1\right) \left(1 - \sqrt{1 - \frac{1}{(\Delta P_1 \sqrt{3} r_\Lambda)^2}}\right) \tag{53}$$

and

$$\Delta P_1 \approx \left(\frac{\Delta X_1}{r_\Lambda^2}\right) \left(1 - \sqrt{1 - \frac{r_\Lambda^2}{(\Delta X_1)^2}}\right) \tag{54}$$

again the figures 1 and 2 under the appropriate normalizations for the cut-offs, represent the behavior for the Cosmological Horizon surface gravity if we take into account that r_{CH} decreases as the mass of the Black Hole increases. It means that the Cosmological Horizon has a positive heat capacity and is stable. If 53 and 54 represent extensions of the same graphic in different regimes, it is again clear that there must be a point at which both became the same. This point would be the UV-IR mix scale for the Cosmological Horizon in agreement with the observations of the "natural" static observer. This new scale is obtained by introducing the condition $\Delta X_1 = \sqrt{3} r_\Lambda^2 \Delta P_1$ inside the GUP expression given in 27. After replacing this condition in 53 and 54, it is trivial to find that the UV-IR scale for the Cosmological Horizon is also of the order of magnitude of r_Λ. It seems then that the S-dS surface gravities corresponding to both, the Black Hole event horizon and the Cosmological one, can be described by the the solutions 48 and 49 if we redefine the UV and IR cut-offs for the Cosmological Horizon case. The solutions 51 and 52 seem to be unnecessary for the description of the S-dS Black Hole from the "natural" static observer point of view.

VIII. THE ANTI-DE SITTER BLACK HOLE TEMPERATURE AND GUP

The Anti-de Sitter space is characterized by a metric written as in eq. 1 but with an opposite sign for Λ. In [15], the Anti-de Sitter Black Hole temperature is given by:

$$T = \frac{1}{4\pi} \frac{r_\Lambda^2 + r_+^2}{r_\Lambda^2 r_+} \tag{55}$$

r_+ is the event horizon for the A-dS Black Hole. In this case, it is trivial to demonstrate that there is only one event horizon which increases with the mass. It is approximately $r_+ \approx 2GM$ as the condition $M << M_{crit}$ is satisfied. M_{crit} is the mass at the Black Hole temperature takes a minimum value given by $T_{min} = \frac{1}{2\pi} \frac{1}{r_\Lambda}$ [15]. This is in fact consistent

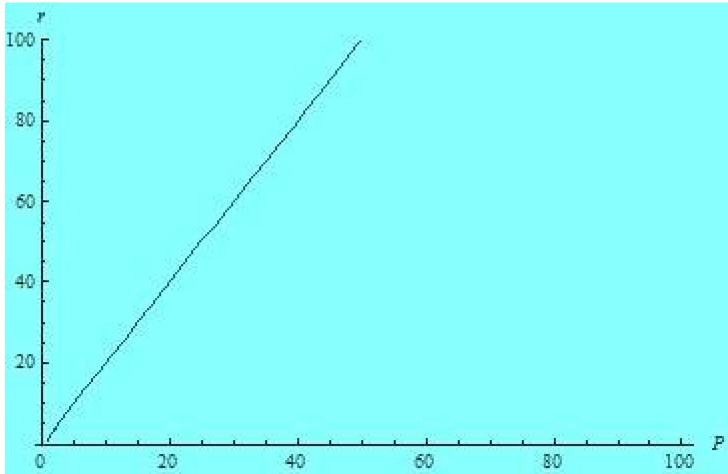

FIG. 3: Distance vs Momentum in agreement with the solution 51. r_Λ has been normalized to unity for convenience. It is evident the positive heat capacity behavior if the event horizon r_+ increases with the mass.

with the result 48 as $(l_{pl}r_\Lambda)^{1/2} \leq \Delta X_1 \leq r_\Lambda$. In 55, we assume $r_\Lambda = \frac{1}{\sqrt{-\Lambda}}$ with $\Lambda < 0$. The relation between the event horizon and the mass of the A-dS Black Hole is given by [15]:

$$M = \frac{1}{2} m_{pl}^2 r_+ \left(1 + \frac{r_+}{3r_\Lambda}\right) \tag{56}$$

M_{crit} can be found as the mass after which the Black Hole temperature increases with the mass, it is simply:

$$M_{crit} = \frac{2}{3}\frac{m_{pl}^2}{m_\Lambda} \tag{57}$$

as $M = M_{crit}$, then $r_+ = r_\Lambda = \frac{1}{m_\Lambda}$. As $M > M_{crit}$, the temperature increases with the mass (positive heat capacity). As $M < M_{crit}$, the temperature decreases with the increasing mass (negative heat capacity). It is clear from 55 that as $r_+ << r_\Lambda$, $T \approx \frac{1}{4\pi r_+}$ which is just the standard Schwarzschild behavior.

The expressions 48 and 49 obtained from GUP describe the appropriate behavior for the A-dS Black Hole as far as $M < M_{max}$ (see the figures 1 and 2), this corresponds to the negative heat capacity region.

On the other hand, as $M > M_{max}$ the temperature increases with the mass. If $r_+ >> r_\Lambda$, then $T \approx \frac{1}{4\pi}\frac{r_+}{r_\Lambda^2}$. If we again accept the analogy $\Delta X_2 \approx r_+$ and $\Delta P_2 \approx \kappa = 2\pi T$, then the equation 51 which is one of the solutions for GUP, can describe the appropriate behavior for the A-dS black Hole as $M > M_{crit}$. Remember that we have already demonstrated that as $\Delta P_2 >> \frac{1}{r_\Lambda}$, the GUP solution 51 is approximately $\Delta P_2 \approx \frac{\Delta X_2}{r_\Lambda^2}$ which has the same linear behavior for A-dS Black Hole in the same regime (see the figure 3).

IX. A DEEP COMPARISON BETWEEN THE S-DS AND S-ADS BLACK HOLE TEMPERATURES

It is interesting to note that the thermodynamic behavior of a S-dS Black Hole is just similar to that obtained for the A-dS one as $M \leq M_{crit}$ if the temperature is measured by the "natural" Static Observer. In both cases, we have a negative heat capacity and the same minimum temperature $\kappa = \sqrt{|\Lambda|}$. We also have the same value for the event horizon as the Holes reach the minimum temperature, they are $r_{BH} = r_+ = r_\Lambda$ as $T \to T_{min} \approx \frac{1}{r_\Lambda}$. This is not a coincidence since both spaces have the same IR cut-off scale although different geometry and causal structure. As far as $M << M_{max}$ and also $M << M_{crit}$, both, the S-dS Black Hole event horizon and the A-dS one, are approximately $r_{BH} \approx 2GM \approx r_+$. In such a case, the conditions $r_{BH} << r_\Lambda$ and $r_+ << r_\Lambda$ are also satisfied [11]. Then the Killing vector normalization in agreement with the "natural" Static Observer position (see equation 13) is just $K^2 \approx -1$ for the S-dS case (see eq. 15) and:

$$\kappa_{BH} = \kappa_{AdS} = \frac{1}{r_{BH,+}} \tag{58}$$

the GUP modifications near the UV regime applies for both cases, S-dS and S-AdS (see eq. 49). If $M << M_{max}$ we can make an expansion for eq. 15. in order to obtain the first order corrections as:

$$\kappa_{dS} \approx \frac{1}{2}\left(1 + \frac{1}{2}\left(\frac{3}{2}\frac{r_s}{r_\Lambda}\right)^{2/3}\right)\frac{1}{r_{BH}} \tag{59}$$

the first order correction is due to the normalization of the Killing vector and the fact that the temperature is measured by the "natural" static observer. This correction increases the effective coefficient of the term proportional to $\frac{1}{r_H}$. In this way, the temperature still decreases with the increasing mass (negative heat capacity) but at a smaller rate in comparison with the standard surface gravity with a Killing vector normalization $K^2 = -1$. We can check what happens in the same regime with the A-dS Black Hole temperature. We can make the same expansion for the A-dS black Hole and compare with 59. If we assume that in the mentioned regime $M << M_{crit}$ (for the A-dS case), then we can assume $r_+ \approx 2GM$ and the temperature 55 would be proportional to:

$$\kappa_{AdS} \approx \frac{1}{2r_+} + \frac{r_+}{2r_\Lambda^2} \tag{60}$$

if we want to compare with 59, we must use the approximations $r_{BH} \approx 2GM$ as $M << M_{max}$ for the S-dS black hole and $r_+ \approx 2GM$ as $M << M_{crit}$ for S-AdS. In such a case, the expressions 59 and 60 become:

$$\kappa_{dS} \approx \frac{1}{4GM} + \frac{9^{1/3}}{8}\frac{1}{(r_\Lambda^2 GM)^{1/3}} \tag{61}$$

and:

$$\kappa_{AdS} \approx \frac{1}{4GM} + \frac{GM}{r_\Lambda^2} \tag{62}$$

the first order corrections obtained in equations 61 and 62 as $M << M_{max} = \frac{1}{2}M_{crit}$ are in general negligible for most of the mass range for both Black Holes (dS and AdS). Now we can expand both temperatures as $r_{BH} \to r_\Lambda$ and $r_+ \to r_\Lambda$. We can assume for example that $r_+ = Cr_\Lambda$, where C is some arbitrary constant near to the unit value. In such a case, 55 can be written as:

$$\kappa_{AdS} \approx \frac{1}{r_\Lambda}\left(\frac{1}{2C} + \frac{C}{2}\right) \tag{63}$$

the expansion for κ_{dS} near κ_{min} has been already performed by Bousso and Hawking. It is given by 16 and repeated here for clarity:

$$\kappa_{dS} \approx \frac{1}{r_\Lambda}\left(1 + \frac{2}{3}\epsilon\right) \tag{64}$$

if we want the two expressions, 63 and 64 to be the same, it is necessary to satisfy the constraint:

$$C = 1 + \frac{2}{3}\epsilon \pm \frac{2}{\sqrt{3}}\sqrt{\epsilon} \tag{65}$$

the negative sign is the relevant to our comparison. The positive sign solution, corresponds to the positive heat capacity region for the AdS Black Hole. We see that as $M = M_{max}$ for S-dS, then $\epsilon \to 0$ and $C = 1$, which is completely consistent. The opposite is also true, if $C = 1$, then $M = M_{crit}$ for the AdS black hole and then $\epsilon \to 0$. Then if the surface gravity for the S-dS Black Hole is defined with respect to the "natural" Static Observer position, then there is a mass region where there is no distinction between the S-dS Black Hole temperature and the AdS one. This fact permits us to obtain a single temperature expression (GUP) for both, de-Sitter and anti-de Sitter space.

X. CONCLUSIONS

In this paper, we have explored the consequences of GUP with an UV cut-off and an IR one. We have demonstrated that the same GUP expression can reproduce the thermodynamic for both; Schwarzschild de-Sitter Black Hole and the Schwarzschild Anti de-Sitter one without making any analytic continuation for the IR cut-off scale. In the case of S-dS, the extension of the analogy with GUP for the Cosmological Horizon temperature is obtained if we redefine the UV and IR cut-offs in agreement with the conditions imposed by the "natural" Static Observer. This implies a redefinition for the minimum uncertainties in position and momentum for the Cosmological Horizon case.
For the S-AdS case, the GUP analogy is natural and it makes use all the solutions obtained from the GUP expression (except the transplanckian modes). GUP can reproduce both, the negative Heat capacity region and the other positive heat capacity one.
Finally we made a comparison between the dS Black Hole and the AdS one in order to analyze their thermodynamic similarities. This is motivated by the fact that it seems that GUP with UV and IR cut-off provides appropriate expressions for both spaces with different geometries and different causal structures. Why is this the case, is an inquiry that will be analyzed in a future manuscript by using the full machinery of the q-Bargmann Fock algebras.

[1] John A. Wheeler *On the nature of Quantum Geometrodynamics*, Ann. Phys. **2**. 604-614 (1957).

[2] C. A. Mead *Possible connection between Gravitation and Fundamental length*, Phys. Rev. **135**, B849 (1964); M. Maggiore *A Generalized Uncertainty Principle in Quantum Gravity*, Phys. Lett. **B 304**, 65 (1993); L. J. Garay *Quantum Gravity and minimum length*, Int. J. Mod. Phys. **A 10**, 145 (1995).

[3] P. Nicolini, *Nonlocal and Generalized Uncertainty Principle Black Holes*, arXiv:1202.2102.

[4] P. Nicolini, *Noncommutative Black Holes, the final appeal to Quantum Gravity: A Review*, Int. J. Mod. Phys. **A24**, (2009), 1229-1308.

[5] I. Arraut, D. Batic and M. Nowakowski, *A Noncommutative model for a Mini Black Hole*, Clas. Quant. Grav. **26** (2009), 245006.

[6] I. Arraut, D. Batic and M. Nowakowski, *Maximal Extension of the Schwarzschild Spacetime inspired by Noncommutative Geometry*, JMP **51** (2010), 022503.

[7] L. N. Chang, Z. Lewis, D. Minic and T. Takeuchi *On the minimal Length Uncertainty Relation and the Foundations of String Theory*, Adv. High Energy Phys. 2011 (2011), 493514.

[8] Pisin Chen, *Generalized Uncertainty Principle and Dark Matter*, astro-ph/0305025; Pisin Chen and Ronald J. Adler, *Black Hole remnants and Dark Matter*, Nucl. Phys. Proc. Suppl. **124** (2003), 103-106; Ronald. J. Adler, Pisin Chen and David I. Santiago, *The Generalized Uncertainty Principle and Black Hole remnants*, Gen. Rel. Grav. **33** (2001) 2101-2108.

[9] Achim Kempf, *On Quantum Field Theory with Nonzero Minimal Uncertainties in Positions and Momenta*, J. Math. Phys. **38** (1997) 1347-1372.

[10] Achim Kempf, *Quantum Field Theory with Nonzero Minimal Uncertainties in Positions and Momenta*, hep-th/9405067.

[11] I. Arraut, D. Batic and M. Nowakowski, *Comparing two approaches to Hawking radiation of Schwarzschild-de Sitter Black Holes*, Class.Quant.Grav. **26** (2009), 125006.

[12] B. Bolen and M. Cavaglia, *(Anti-) de Sitter Black Hole Thermodynamics and the Generalized Uncertainty Principle*, Gen. Rel. Grav. **37** (2005), 1255.

[13] M. Nowakowski and I. Arraut, *The Minimum and Maximum temperature of Black Body Radiation*, Mod. Phys. Lett. **A 24** (2009), 2133.

[14] R. Bousso and S. W. Hawking, *Pair creation of black holes during inflation*, Phys. Rev. **D 54** (1996), 6312-6322.

[15] S. W. Hawking and D. N. Page, *Thermodynamics of Black Holes in Anti-de Sitter Space*, Commun. Math. Phys **87**, 577, (1983).

[16] S. L. Bażński and V. Ferrari, *Analytic Extension of the Schwarzschild-de Sitter Metric*, Il Nuovo Cimento Vol. 91 B, N. 1, 11 Gennaio (1986).

[17] R. Bousso and S. W. Hawking, *(Anti)-evaporation of Schwarzschild-de Sitter black holes*, Phys. Rev. **D 57** (1998), 2436-2442.

[18] P. Ginsparg and M. J. Perry, Nucl. Phys. **B222**, 245 (1983).

[19] L. C. Barbado, C. Barcel and L. J. Garay, *Hawking radiation as perceived by different observers*, Class.Quant.Grav. **28** (2011) 125021; L.C. Barbado, C. Barcelo and L. J. Garay, *Hawking radiation as perceived by different observers: an analytic expression for the effective-temperature function*, Class.Quant.Grav. **29** (2012) 075013.

[20] A. Kempf *Quantum Group Symmetric Fock Spaces with Bargmann-Fock representations*, Lett. Math. Phys. **26**:1-12 (1992).